Journal of Statistical Physics, Vol. 88, Nos. 1/2, 1997

Statistical Mechanics of Braided Markov Chains: I. Analytic Methods and Numerical Simulations

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Received April 30, 1996; final October 30, 1996

We investigate numerically and analytically the statistics of Markov chains on so-called braid (B_n) and locally free ($\mathscr{LF}_n$) groups. Namely, we compute the mean length $\langle \mu \rangle$ and the variance $\langle \mu^2 \rangle - \langle \mu \rangle^2$ of the shortest word which remains after applying of all group relations to the randomly generated N-letter word (Markov chain). We express the conjecture (numerically justified) that the mean value $\langle \mu \rangle$ for the random walk on the group B_n ($n \gg 1$) coincides with high accuracy with the same value for the random walk on the "locally free group with errors" if the number of errors is of order of 20%.

KEY WORDS: Random walk; braid group; graph of the group; primitive word; symbolic dynamics.

INTRODUCTION

Considerable number of works is devoted to investigation of the liquid-crystalline-type phase transitions in systems of long chain molecules (for review see for instance refs. 1 and 2). Apparently, at present the scope of problems dealing with the nematic-type ordering in polymers is one of the most examined branches of statistical physics of macromolecules. However, as far as we know, all the existing theories do not take into account the effects caused by entanglements between the chains in such systems.

In the present work we develop the basis of the simple mean-field theory of ordering phase transition in the system of entangled "directed polymers," i.e., in the "braid" of fluctuating polymer chains with fixed topology in an external field. Let us stress from very beginning that we do

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0022-4715/97/0700-0201$12.50/0

not claim to find the new kind of phase transitions or to describe the new class of real physical systems. The main goal of our investigation concerns the construction of the mean-field-like theory of fluctuating entangled chains in $1+1$ dimensions utilizing our knowledges elaborated in course of investigation of statistics of Markov chains on the braid and locally free groups. The forthcomming publication[3] (hereafter reffered as II) will be devoted to examination of the influence of topological constraints on the standard nematic-like phase transition in bunch of "braided polymers."

Topological constraints essentially modify the physical properties of statistical systems consisting of chain-like objects of completely different nature. It should be said that topological problems are widely investigated in connection with quantum field and string theories, 2D-gravitation, statistics of vortices in superconductors, quantum Hall effect, thermodynamic properties of entangled polymers etc. Modern methods of theoretical physics allow us to describe rather comprehensively the effects of nonabelian statistics on physical behavior for each particular referred system; however, in our opinion, the following general question remain obscure:

(a) How does the changes in topological state of the system of entangled chain-like objects effect their physical properties?

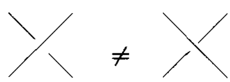
(b) How can the knowledge accrued in statistical topology be applied to the construction of the Ginzburg–Landau-type theory of fluctuating entangled (nonabelian) chain-like objects?

In order to have representative and physically clear image for the system of fluctuating chains with the full range of nonabelian topological properties it appears quite natural to formulate general topological problems in terms of polymer physics. It allows us: (i) to use a geometrically clear image of polymer with topological constraints as a model corresponding to the path integral formalism in the field theory; (ii) to advance in investigation of specific physical properties of biological and synthetical polymer systems where the topological constraints play a significant role.

This paper mainly concerns the general problems dealing with the topological properties of polymers represented as Brownian trajectories. The connection between topology and detailed chemical structure of polymers as well as the statistical properties of macromolecules caused by their specific chemical structure are beyond the scope of the present book.

For physicists the polymer objects are attractive due to many reasons. First of all, the adjoining of monomer units in chains essentially reduces all equilibrium and dynamic properties of the system under consideration. Moreover, due to that adjoining the behavior of polymers is determined by the space-time scales larger than for low-molecular-weight substances. This allows us to apply general theoretical methods such as perturbation theory, renormalization-group approach, conformal methods etc. to the investigation of polymer systems consisting of both ensemble of chains and a single macromolecule. Major progress in theoretical description of polymer systems is due to the combination of general methods of solid-state physics with the methods which take into account the chain-like structure of polymers. The chain-like structure of macromolecules causes the following peculiarities (see, for instance, ref. 4): (i) the so-called "linear memory" (i.e., fixed position of each monomer unit along the chain); (ii) the low translational entropy (i.e., the restrictions on independent motion of monomer units due to the presence of bonds); (iii) the large space fluctuations (i.e., just a single macromolecule can be regarded as a statistical system with many degrees of freedom).

It should be emphasized that the above mentioned "linear memory" leads to the fact that different parts of polymer molecules fluctuating in space can not go one through another without the chain rupture. For the system of non-phantom closed chains this means that only those chain conformations are available which can be transformed continuously into one another:



what inevitably give rise to the problem of knot entropy determination.[5]

After these preliminary remarks it becomes clear that many general (noncommutative) topological problems dealing with statistics of chain-like objects can be formulated easily in polymer language.

Thus, the reason of our investigations is forced by real topological problems which we are going to describe in more details in the part II of the work. Here let us just mention that the scope of tasks dealing with the nematic-type ordering in bunches of entangled polymers as well as consideration of thermodynamic properties of uncrossible vortex lines immediately turn us to studying of statistics of chain-like objects with nonabelian topology.

Problems dealing with the limit distributions of random walks on some noncommutative groups is represented rather widely in the probability theory. Namely, the set of rigorous results concerning the limit behavior of Markov chains on the free group and on the Riemann surfaces of constant negative curvature have been received in works;[6–9] the problem of construction of the probability measure for random walks on the modular group $PSL(2, \mathbb{Z})$ has been studied in ref. 10. To this theme we could attribute also number of spectral problems considered in the theory of dynamic systems on hyperbolic manifolds[11, 12] as well as the subject of the random matrix theory.[13]

However in the context of "topologically-probabilistic" consideration, the limit distributions of noncommutative random walks are practically out of discussion except very few specific cases.[14-16] In particular, in these works it has been shown that statistics of random walks with fixed topological state with respect to the regular array of obstacles on the plane can be obtained from the limit distributions of the so-called "brownian bridges" (see the definition below) on the universal covering—the graph with topology of the Cayley tree. The analytic construction of the non-abelian topological invariants for the trajectories on the double punctured plane and statistics of simplest nontrivial random braid B_3 was shortly discussed in ref. 17.

Our main goal of the present work is as follows: we consider analytically and numerically the limit behavior of the Markov chains where the states are randomly taken from some noncommutative finite discrete group deeply related to topology. In particular, we restrict ourselves with the so-called *braid* (B_n) and *locally free*[3] ($\mathscr{LF}_n$) groups—see the definitions in the next section. The first brief combinatorial analysis of locally free groups was undertaken in recent works.[9]

1. BASIC DEFINITIONS AND MODEL

1.1. Random Walks over Group Elements

We begin with the investigation of the probabilistic properties of Markov chains on simplest noncommutative groups. In the most general way the problem can be formulated as follows.

Take a discrete group $\mathscr{G}_n$ with fixed finite number of generators $\{g_1,..., g_{n-1}\}$. Any arbitrary ordered sequence of generators we call the *word*. The *length* of the word, N, is the total number of used generators ("letters"), whereas the *minimal irreducible length*, μ, called below the "primitive word" is the shortest noncontractible length of a particular word which remains after applying of all possible group relations.

Let ν be the uniform distribution on the set $\{g_1,..., g_{n-1}, g_1^{-1},..., g_{n-1}^{-1}\}$. For convenience we take $h_j = g_i$ for $j = i$ and $h_j \equiv g_i^{-1}$ for $j \equiv i+n-1$; $\nu(h_j) = 1/2n-2$ for any j. We construct the (right-hand) random walk (the random word) on $\mathscr{G}_n$ with a transition measure, ν, i.e., $\{\xi_n\}$; $\xi_0 = e \in \mathscr{G}_n$ and $\mathrm{Prob}(\xi_j = u \mid \xi_{j-1} = v) = \nu(v^{-1}u) = 1/2n-2$. It means that with the probability

[3] The notation "locally free group" is proposed by A. M. Vershik.

$1/2n-2$ we add the element h_{α_N} to the given word $h_{N-1}=h_{\alpha_1}h_{\alpha_2}\cdots h_{\alpha_{N-1}}$ from the right-hand side.[4]

The random word W formed by N letters taken independently with the uniform probability distribution $v=1/2n-2$ from the set $\{g_1,..., g_{n-1}, g_1^{-1},..., g_{n-1}^{-1}\}$ is called the *Markov chain* of length N on the group $\mathscr{G}_n$.

The most attention is paid in this paper to the following question: What is the number of possibilities, $Z(\mu, N)$, to reduce all N-letter words to the primitive word of length μ. The quantity $Z(\mu, N)$ plays a role of the partition function for the random walk on the group $\mathscr{G}_n$ with the uniform probability distribution v over group generators.

1.2. Braid and "Locally Free" Groups

We are aimed to study the asymptotics of the limit distributions of Markov chains on the braid group B_n. For the case $n=3$ the problem has been solved in ref. 9, where the limit probability distribution as well as the conditional limit probability distribution of "brownian bridges" on the group B_3 has been derived. For $n>3$ this problem is unsolved yet. However we can extract some estimations for the limit behavior of Markov chains on B_n considering the random walks on so-called "locally free groups."[(7, 9)]

Braid Group. The braid group B_n of n strings has $n-1$ generators $\{\sigma_1, \sigma_2,..., \sigma_{n-1}\}$ with the following relations:

$$\begin{aligned} \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && (1 \leqslant i < n-1) \\ \sigma_i\sigma_j &= \sigma_j\sigma_i && (|i-j| \geqslant 2) \\ \sigma_i\sigma_i^{-1} &= \sigma_i^{-1}\sigma_i = e \end{aligned} \tag{1}$$

Let us mention that:

— The word written in terms of "letters"—generators from the set $\{\sigma_1,..., \sigma_{n-1}, \sigma_1^{-1},..., \sigma_{n-1}^{-1}\}$ gives a particular *braid*—see Fig. 1a.

— The closed braid appears after gluing the "upper" and the "lower" free ends of the braid on the cylinder (Fig. 1b).

— Any braid corresponds to some knot or link. So, there is a principal possibility to use the braid group representation for the construction of topological invariants of knots and links, but the correspondence of braids and knots is not mutually single valued and each knot or link can be represented by infinite series of different braids. This fact should be taken into account in course of the knot invariants construction.

[4] Analogously we can construct the left-hand side random walk on the group $\mathscr{G}$.

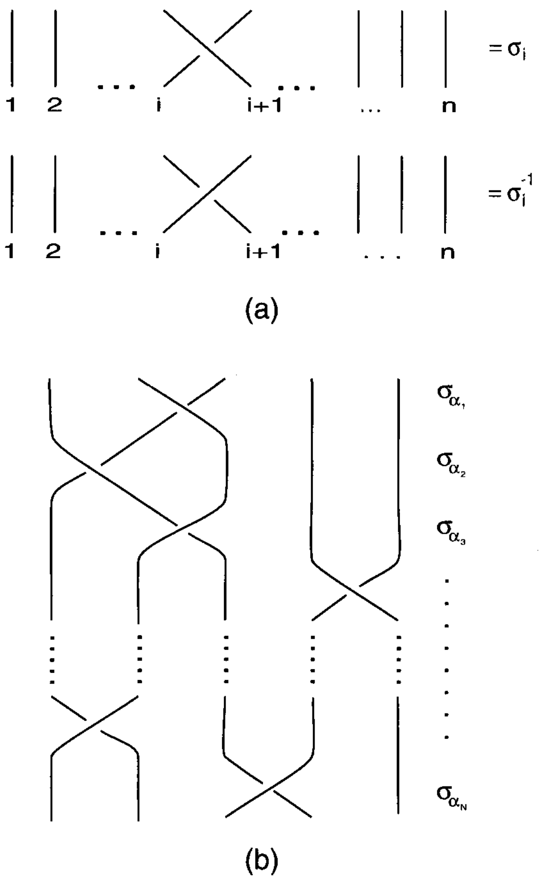


Fig. 1. (a) Geometric representation of generators σ_i ("positive") and σ_i^{-1} ("negative") in the group B_n; (b) Schematic representation of a particular braid of N generators.

Locally Free Group. The group $\mathscr{LF}_n(d)$ is called the *locally free* if generators, $\{f_i,..., f_{n-1}\}$ obey the following commutation relations:

(a) Each pair (f_j, f_k) generates the free subgroup of the group $\mathscr{LF}_n$ if $|j-k| < d$;

(b) $f_j f_k = f_k f_j$ for $|j-k| \geqslant d$

We pay the most attention to the case $d = 2$ for which we define $\mathscr{LF}_n(2) \equiv \mathscr{LF}_n$.

It can be seen that the only one difference between the braid and locally free groups consists in the replacement of the Yang–Baxter relations

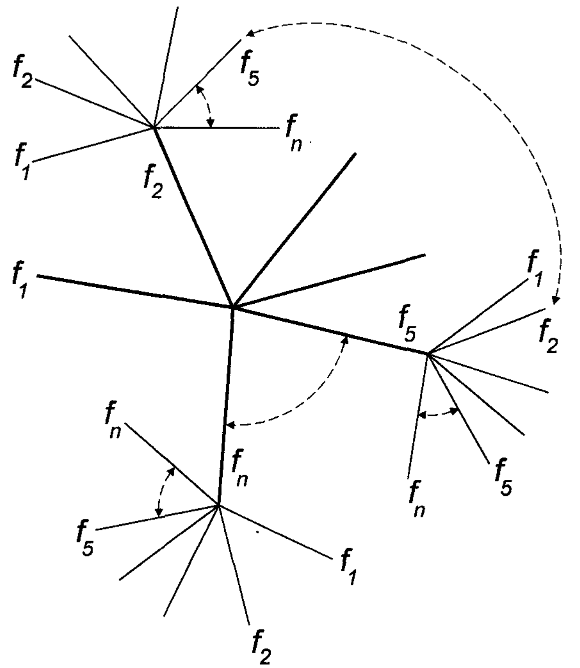


Fig. 2. Vertices corresponding to the words $f_2 f_5$ and $f_5 f_2$ should be glued because they represent one and the same word in the group $\mathscr{LF}_{n+1}$.

(in case of B_n) by relations of the free group (in case of $\mathscr{LF}_n$). In Fig. 2 the graph $C(\mathscr{LF}_n)$ corresponding to the group $\mathscr{LF}_n$ is shown schematically where the vertices corresponding to the equivalent primitive words should be identified. (Two words are equivalent if they can be transformed into one another by permutations allowed in by the definition of the group).

2. STATISTICS OF RANDOM WALKS ON BRAID AND LOCALLY FREE GROUPS

It has been shown in papers[14] that for the free group (i.e., for the group without any commutation relations among generators) the problem about the limit distribution of Markov chains can be mapped to the investigation of random walks on a simply connected tree. In case of braids the more complicated group structure does not allow us to use the simple geometrical image directly. Nevertheless the problem of the limit distribution of random walks off B_n can be reduced to the consideration of the random walk on some graph. In the case of the group B_3 we are able to construct this graph evidently, while for the group B_n ($n \geqslant 4$) we can give an estimation for the limit distribution of random walks considering the statistics of Markov chains on so-called local groups (see ref. 9 and below).

2.1. Numerical Results

The goal of our numerical computations consists in comparison of the expectation values for the random walk on braid group B_n and on locally free group $\mathscr{L}\mathscr{F}_n$ (see the definitions above).

We believe that our results will give some insight to the problem of random walk in the noncommutative groups related to topology (like braid group) and will stimulate the forthcomming investigations. The reason of the replacement of the braid group by the locally free one is due to the fact that the problem of limit distribution of Markov chains on locally free group admits the exact consideration (see also Section 3).

The model used for numerical simulations has been explained in Section 1.2. Let us point out the main steps of our computations:

(a) We generate randomly (with uniform probability distribution) the words of lengths $N \in [1000; 20000]$, while the number of generators, n, varies in the interval $[3; 200]$. The number of randomly generated words is of order of 1000.

(b) We reduce the given word till the minimal irreducible (primitive) word. This can be done by using the braid (or locally free) group relations. The numerical procedure is as follows. First, we try to push each braid generator in the word as far as possible to the left. Some reductions can occur after that. Then, we play the same game but in the opposite direction, pushing each braid generator to the right performing possible reductions of the word, then—to the left again and so on.... If no reductions occur during two consecutive steps, we stop the process.

We compute the following quantities for braid and locally free groups:

The mean length of the shortest (primitive) word $\langle \mu \rangle$

$$\langle \mu \rangle = \frac{\sum_{\mu=0}^{\infty} \mu Z(\mu, N)}{\sum_{\mu=0}^{\infty} Z(\mu, N)} \tag{2}$$

and the variance $\mathrm{Var}(\mu)$

$$\mathrm{Var}(\mu) \equiv \langle \mu^2 \rangle - \langle \mu \rangle^2 = \frac{\sum_{\mu=0}^{\infty} \mu^2 Z(\mu, N)}{\sum_{\mu=0}^{\infty} Z(\mu, N)} - \langle \mu \rangle^2 \tag{3}$$

The results of numerical simulations for the word statistics on braid (B_n) and locally free ($\mathscr{L}\mathscr{F}_n(d)$) groups are presented in the Table 1.

Table 1[a]

Groups	B_n $\frac{\langle\mu\rangle}{N}$	B_n $\frac{\mathrm{Var}(\mu)}{N}$	$\mathscr{LF}_n(2)$ $\frac{\langle\mu\rangle}{N}$	$\mathscr{LF}_n(2)$ $\frac{\mathrm{Var}(\mu)}{N}$	$\mathscr{LF}_n(3)$ $\frac{\langle\mu\rangle}{N}$	$\mathscr{LF}_n(3)$ $\frac{\mathrm{Var}(\mu)}{N}$	$\mathscr{LF}_n(4)$ $\frac{\langle\mu\rangle}{N}$	$\mathscr{LF}_n(4)$ $\frac{\mathrm{Var}(\mu)}{N}$
$n=3$	0.29	0.85	0.50	0.76	0.50	0.76	0.50	0.75
$n=5$	0.49	0.77	0.60	0.63	0.71	0.48	0.75	0.46
$n=10$	0.56	0.63	0.65	0.56	0.77	0.40	0.82	0.34
$n=20$	0.59	0.63	0.66	0.54	0.79	0.39	0.84	0.29
$n=50$	0.61	0.61	0.67	0.56	0.80	0.38	0.85	0.27
$n=100$	0.61	0.61	0.67	0.52	0.80	0.36	0.86	0.26
$n=200$	0.61	0.60	0.67	0.53	0.80	0.35	0.86	0.26

[a] The groups $\mathscr{LF}_n(d)$ are completely free when $d \geqslant n-1$.

The maximal standard deviations in the Table 1 (and everywhere below) are:

$$\begin{cases} \pm 0.01 \text{ for the mean value } \langle\mu\rangle/N \\ \pm 0.05 \text{ for the variance } \mathrm{Var}(\mu)/N \end{cases}$$

2.2. Analytic Results for Random Walks on Locally Free Group

Let us estimate now the quantities $\langle\mu\rangle/N$ and $\mathrm{Var}(\mu)/N$ analytically. We present below two different approaches called "dynamical" and "statistical." The "dynamical" approach is based on simple estimation of the probability to reduce the primitive word by random adding one extra letter. The estimate obtained by this method is in very good agreement with corresponding numerical simulations. However the "statistical" approach dealing with rigorous enumeration of all nonequivalent primitive words in the locally free group $\mathscr{LF}_n(d)$ leads to another answer. The rest of this section is devoted to the explanation of the above-mentioned discrepancy.

Dynamical Consideration. Under the conditions

$$\begin{aligned} &n \gg 1 \\ &N \gg n^2 \end{aligned} \tag{4}$$

we can easily develop the dynamical arguments which are in rather good agreement with the results of numerical simulations presented above. The last inequality in ref. 4 ensures the conditions, sufficient for finding the limit probability distribution of Markov chains on the groups of n generators. Actually, the number of letters in the word, N should be much larger that the number of all possible pairs in the set of $2n$ letters.[5] Only in this case the corresponding Markov process has the reliable distribution function. The number of pairs is of order $4n^2$, so we arrived at the inequality stated in ref. 4.

Take a randomly generated N-letter word W. This word is characterized by the length of the primitive word W_p (recall that W_p is the length of the word W obtained after all possible contractions allowed by the structure of the group $\mathscr{L}\mathscr{F}_n(d)$).[6]

Let us compute the probability $\pi(d)$ of the fact that the primitive word W_p will be shortened in one letter after adding of the letter $f_i (i \in [1, n])$ to the word W from the right-hand side. It is easy to understand that the primitive word W_p can be reduced if:

(a) The last letter in the word W_p is just f_i^{-1}. The probability of such event is $1/2n$;

(b) The letter before the last in the word W_p is f_i^{-1} and the last letter commutes with the letter f_i. The probability of such event is $1/2n(1-(4d-2)/2n)$;

(c) The third letter from the right end of the word W_p is f_i^{-1} and two last letters commute with the letter f_i. The probability of such event is $1/2n(1-(4d-2)/2n)^2$;

(d) ... and so on.

Finally we arrive at the following expression for the probability $\pi(d)$:

$$\pi(d) = \frac{1}{2n} \sum_{l=0}^{\infty} \left(1 - \frac{4d-2}{2n}\right)^l = \frac{1}{4d-2} \tag{5}$$

The procedure described above assumes that the letters remaining in the word W_p are uniformly distributed—as in the initial (nonreduced word W). The absence of "boundary effects" is ensured by the condition (4).

Once having the probability $\pi(d)$, we can write down the master equation for the probability $P(\mu, N)$ of the fact that in randomly generated N-letter word the primitive path has the length μ

$$P(\mu, N+1) = (1-\pi(d))\, P(\mu-1, N) + \pi(d)\, P(\mu+1, N) \quad (\mu \geqslant 2) \tag{6}$$

[5] The total number of generators is $2n$ because each of n generators has the inverse one.
[6] Our consideration is valid for any values of d.

where the relation between $P(\mu, N)$ and the partition function $Z(\mu, N)$ introduced above is as follows

$$P(\mu, N)=\frac{Z(\mu, N)}{\sum_{\mu=0}^{\infty} Z(v, N)}$$

The recursion relation (6) coincides with the equation describing the random walk on the halfline with the drift from the origin or, what is the same, with the equation describing the random walk on the simply Cayley tree with the coordination number

$$z_{\text{eff}}=\frac{1}{\pi(d)}=4d-2 \tag{7}$$

Taking into account the last analogy we can complete the Eq. (6) by the boundary conditions

$$\begin{aligned}
P(\mu=1, N+1) &= P(\mu=0, N)+\pi(d)\, P(\mu=2, N) \\
P(\mu=0, N+1) &= \pi P(\mu=1, N) \\
P(\mu, N=0) &= \delta_{\mu, 0}
\end{aligned} \tag{8}$$

It is noteworthy that these equations are written just for the Cayley tree with z_{eff} branches. The actual structure of the graph corresponding to the group $\mathscr{L}\mathscr{F}_n(d)$ is much more complex, thus Eqs. (8) should be regarded as an approximation. However the exact form of boundary conditions does not influence the asymptotic solution of Eq. (6) in vicinity of the maximum of the distribution function:

$$P(\mu, N) \simeq \frac{1}{2\sqrt{2\pi(z_{\text{eff}}-1)\, N}} \exp\left\{-\frac{z_{\text{eff}}^2}{8(z_{\text{eff}}-1)\, N}\left(\mu-\frac{z_{\text{eff}}-2}{z_{\text{eff}}} N\right)^2\right\} \tag{9}$$

Thus, we find

$$\begin{aligned}
\langle \mu(d) \rangle &\simeq \frac{z_{\text{eff}}-2}{z_{\text{eff}}}=\frac{2d-2}{2d-1} \\
\frac{\operatorname{Var}(\mu, d)}{N} &\simeq \frac{4(z_{\text{eff}}-1)}{z_{\text{eff}}^2}=\frac{4d-3}{(2d-1)^2}
\end{aligned} \tag{10}$$

Substituting in Eq. (10) $d=2, 3, 4$, we get the following numerical values:

$$\frac{\langle \mu(d) \rangle}{N} = \frac{2}{3}; \qquad \frac{\mathrm{Var}(\mu, d)}{N} = \frac{5}{9} \qquad \text{for} \quad d=2$$

$$\frac{\langle \mu(d) \rangle}{N} = \frac{4}{5}; \qquad \frac{\mathrm{Var}(\mu, d)}{N} = \frac{9}{25} \qquad \text{for} \quad d=3$$

$$\frac{\langle \mu(d) \rangle}{N} = \frac{6}{7}; \qquad \frac{\mathrm{Var}(\mu, d)}{N} = \frac{13}{49} \qquad \text{for} \quad d=4$$

what is in the excellent agreement with the asymptotic values ($n \gg 1$) from the Table 1 for the same groups.

Statistical Consideration. For the group $\mathscr{LF}_{n+1}(d)$ we can extract the limit behavior of the distribution function $P(\mu, N)$ exactly evaluating the number $V_n(\mu, d)$ of all nonequivalent primitive words of length μ in the group $\mathscr{LF}_{n+1}(d)$. We derive the explicit expression for the function $V_n(\mu, d)$ and show that it has the following asymptotics for $d=2$ and $\mu \gg 1$

$$V_n(\mu, d=2) = \mathrm{const}\left(7 - \frac{8\pi^2}{n^2}\right)^{\mu} \qquad (n \gg 1) \tag{11}$$

To compute $V_n(\mu, d)$ we represent each primitive word W_p of length μ in the group $\mathscr{LF}_{n+1}(d)$ in the **normal order** similar to so-called "symbolic dynamics" used in consideration of chaotic systems (see, for instance, ref. 18)

$$W_p = (f_{\alpha_1})^{m_1} (f_{\alpha_2})^{m_2} \cdots (f_{\alpha_s})^{m_s} \tag{12}$$

where $\sum_{i=1}^{s} |m_i| = \mu$ ($m_i \neq 0$ $\forall i$; $1 \leqslant s \leqslant \mu$) and the sequence of generators f_{α_i} in Eq. (12) **for all distinct** f_{α_i}, satisfies the following local rules in ref. 9:

(i) If $f_{\alpha_i} = f_1$, then $f_{\alpha_{i+1}} \in \{f_2, f_3, \dots, f_n\}$;

(ii) If $f_{\alpha_i} = f_k$ $(1 < k \leqslant n-1)$, then $f_{\alpha_{i+1}} \in \{f_{k-d+1}, \dots, f_{k-1}, f_{k+1}, \dots, f_n\}$;

(iii) If $f_{\alpha_i} = f_n$, then $f_{\alpha_{i+1}} \in \{f_{k-d+1}, \dots, f_{n-1}\}$.

These local rules give the prescription how to encode and enumerate all distinct primitive words in the group $\mathscr{LF}_{n+1}(d)$. If the sequence of generators in the primitive word W_p does not satisfy the rules (i)–(iii), we commute the generators in the word W_p up the normal order is restored. Hence, the normal order representation enables one to give the unique coding of all nonequivalent primitive words in the group $\mathscr{LF}_{n+1}(d)$.

Example 1. Take an arbitrary primitive word of length $\mu = 10$ in the group $\mathscr{L}\mathscr{F}_{8+1}(2)$:

$$W_p = f_5^{-1} f_3 f_8 f_1^{-1} f_2 f_4 f_8 f_8 f_4 f_7 \tag{13}$$

To represent the word W_p in the "normal order" we have to push all generators with smaller indices to the left when it is allowed by the commutation relations of the locally free group $\mathscr{L}\mathscr{F}_9(2)$. We get:

$$W_p = (f_1)^{-1} (f_3)^1 (f_2)^1 (f_5)^{-1} (f_4)^2 (f_8)^3 (f_7)^1 \tag{14}$$

(the "normal order" for this word is the sequence of used generators: $\{1, 3, 2, 5, 4, 8, 7\}$). To compute the number of different primitive words of length $\mu = 10$ with the same normal order as in Eq. (14), we have to sum up all the words like

$$W_p = (f_1)^{m_1} (f_3)^{m_2} (f_2)^{m_3} (f_5)^{m_4} (f_4)^{m_5} (f_8)^{m_6} (f_7)^{m_7} \tag{15}$$

under the condition $\sum_{i=1}^{7} |m_i| = 10$; $m_i \neq 0\ \forall m_i \in [1, 7]$.

The calculation of the number of distinct primitive words, $V_n(\mu)$, of the given length μ is now rather straightforward:

$$V_n(\mu, d) = \sum_{s=1}^{\mu} R_n(s, d) \sum_{\{m_1, \ldots, m_s\}}{}' \Delta\left[\sum_{i=1}^{s} |m_i| - \mu\right] \tag{16}$$

where $R_n(s, d)$ is the number of all distinct sequences of s generators taken from the set $\{f_1, \ldots, f_n\}$ and satisfying the local rules (i)–(iii) while the second sum gives the number of all possible representations of the primitive path of length μ **for the fixed sequence of generators**—(see the example above); "prime" means that the sum runs over all $m_i \neq 0$ for $1 \leqslant i \leqslant s$; Δ is the Kronecker function.

To get the partition function $R_n(s, d)$ let us mention that the local rules (i)–(iii) define the generalized Markov chain with the states given by the $n \times n$ "coincidence" matrix $\hat{T}_n(d)$ where the rows and columns correspond to the generators $f_1, \ldots, f_n$ as it is shown below:

$$\hat{T}_n(d) = \begin{array}{c|cccccccc}
 & f_1 & f_2 & f_3 & f_4 & \cdots & f_{n-1} & f_n \\
\hline
f_1 & 0 & 1 & 1 & 1 & \cdots & 1 & 1 \\
f_2 & 1 & 0 & 1 & 1 & \cdots & 1 & 1 \\
f_3 & 1 & 1 & 0 & 1 & \cdots & 1 & 1 \\
f_4 & 0 & 1 & 1 & 0 & \cdots & 1 & 1 \\
\vdots & \vdots & \vdots & & \ddots & \ddots & \vdots & \vdots \\
f_{n-1} & 0 & 0 & 0 & 0 & \ddots & 0 & 1 \\
f_n & 0 & 0 & 0 & 0 & & 1 & 0
\end{array} \tag{17}$$

The matrix $\hat{T}_n(d)$ has rather simple structure: above the diagonal we put everywhere "1" and below diagonal we have $d-1$ subdiagonals completely filled by "1"; in all other places we have "0" (in Eq. (17) it is shown the case with $d=3$).

The number of all distinct normally ordered **sequences of words** of length s with allowed commutation relations is given by the following partition function

$$R_n(s, d) = \mathbf{v}_{\text{in}}[\hat{T}_n(d)]^s \, \mathbf{v}_{\text{out}} \tag{18}$$

where

$$\mathbf{v}_{\text{in}} = (\overbrace{1 \ \ 1 \ \ 1 \cdots 1}^{n}) \qquad \text{and} \qquad \mathbf{v}_{\text{out}} = \left.\begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}\right\} n \tag{19}$$

The remaining sum in Eq. (16) is independent on $R_n(s, d)$, so its calculation is trivial:

$$\sum_{\{m_1, \ldots, m_s\}}{}' \Delta\left[\sum_{i=1}^{s} |m_i| - \mu\right] = 2^s \frac{(\mu-1)!}{(s-1)!\,(\mu-s)!} \tag{20}$$

Substituting Eq. (20) and Eq. (18) into Eq. (16) we get

$$V_n(\mu, d) = 2n + \sum_{s=1}^{\mu} 2^s \frac{(\mu-1)!}{(s-1)!\,(\mu-s)!} R_n(s, d) \tag{21}$$

The value $V_n(\mu, d)$ is growing exponentially fast with μ and the "speed" of this grows is clearly represented by the fraction

$$q(d) = \left.\frac{V_n(\mu+1), d)}{V_n(\mu, d)}\right|_{\mu \gg 1} \tag{22}$$

We can find the closed asymptotic expression for the value $q(d)$ supposing that the main contribution in Eq. (21) appears from $s \gg 1$. In this case we have for $R_n(s, d)$:

$$R_n(s, d)|_{s \gg 1} = [\lambda_n^{\max}(d)]^s \tag{23}$$

where $\lambda_n^{\max}$ is the highest eigenvalue of the matrix $\hat{T}_n(d)(n \gg 1)$.

In the two limiting cases $d=2(n \gg 1)$ and $d=n-1$ (n is arbitrary) the computations are rather straightforward:

(a) For $d=2$ (see Appendix A for more details) we find the value of the highest eigenvalue $\lambda_n^{\max}(2)$ $(n \gg 1)$ in the form

$$\lambda_n^{\max}(2)|_{n \gg 1} = 3 - \frac{4\pi^2}{n^2} + o\left(\frac{1}{n^2}\right) \tag{24}$$

(b) For $d=n-1$ we get

$$\lambda_n^{\max}(n) = n-1 \tag{25}$$

Example 2. For $n=3$ and $d=2$ we have the free group without any commutation relations, i.e., the Cayley tree with $2(n-1)=4$ branches.

Substituting Eqs. (24)–(25) into Eq. (21) and evaluating the remaining sums over s we obtain

$$\begin{aligned} q(2) &\simeq 7 - \frac{8\pi^2}{n^2} \\ q(n) &= 2n-1 \end{aligned} \tag{26}$$

Let us pay attention to the geometric sense of Eq. (26). First of all recall that the complete free group $\Gamma_n \equiv \mathscr{L}\mathscr{F}_{n+1}(d=n-1)$ (without any commutation relations between the generators) has the structure of z-branching Cayley, where

$$z = q(n) + 1 = 2n$$

what coincides with the well known result that the number of distinct primitive words of length μ in the group Γ_n grows like

$$V_n(\mu, d=n) = [q(n)+1][q(n)]^{\mu-1} = \frac{2n}{2n-1}(2n-1)^{\mu} \tag{27}$$

Comparing Eqs. (11) and (27) we can conclude that the graph corresponding to the locally free group, $\mathscr{L}\mathscr{F}_{n+1}(2)$, can be effectively regarded as z-branching Cayley tree, where

$$z|_{n \gg 1} \simeq q(2) + 1 = 8 - \frac{8\pi^2}{n^2} \tag{28}$$

Despite the local structure of the group $\mathscr{L}\mathscr{F}_{n+1}(d)$ is very complex, Eq. (28) enables us to find the asymptotics of the distribution function

Table 2a. Group $\mathcal{L}\mathcal{F}_n(3)$ $(n \gg \nu \gg 1)$

	z_{eff}	$\langle\mu\rangle/N$	$\text{Var}(\mu)/N$	z	$\langle\mu\rangle/N$	$\text{Var}(\mu)/N$	z_{cor} (see Eq. (33))
$n=20$	6	0.67	0.56	7.8	0.74	0.45	6
$n=50$	6	0.67	0.56	7.9	0.75	0.44	6
$n=100$	6	0.67	0.56	8	0.75	0.44	6

$P_d(\mu, N)$ for the random walk on the group in the same way as it was described in course of derivation of Eq. (9). Thus, to find the expectation values $\langle\mu(d)\rangle/N$ and $\text{Var}(\mu, d)/N$ we have to replace z_{eff} by the value $z = q(d) + 1$. Hence, we get

$$\begin{aligned} \frac{\langle\mu(d)\rangle}{N} &\simeq \frac{z-2}{z} \\ \frac{\text{Var}(\mu, d)}{N} &\simeq \frac{4(z-1)}{z^2} \end{aligned} \tag{29}$$

In the Tables 2a, b we compare the values of z_{eff} with the values $z = q(d) + 1$ and z_{cor} as well as corresponding expectation values $\langle\mu(d)\rangle$ and $\text{Var}(\mu, d)$ for the groups $\mathcal{L}\mathcal{F}_{n+1}(2)$ and $\mathcal{L}\mathcal{F}_{n+1}(3)$ extracted from our analytical results for $n \gg 1$.

We pay attention to the fact that the expectation values computed on the basis of dynamical consideration do not coincide with the values computed from the enumeration of all nonequivalent primitive words in the corresponding group. Some arguments concerning the ways of possible resolution of this contradiction we present below.

Why the coordinational numbers z_{eff} and z are different? The natural question appears now: which Eq.—(7) or (28)—is correct... or

Table 2b. Group $\mathcal{L}\mathcal{F}_n(3)$ $(n \gg 1)$

	z_{eff}	$\langle\mu\rangle/N$	$\text{Var}(\mu)/N$	z	$\langle\mu\rangle/N$	$\text{Var}(\mu)/N$	z_{cor} (see Eq. (33))
$n=20$	10	0.8	0.36	12.5	0.84	0.29	10.1
$n=50$	10	0.8	0.36	12.5	0.84	0.29	10.1
$n=100$	10	0.8	0.36	12.5	0.84	0.29	10.1

both are wrong? Our opinion is that the answer should be: "both Eqs. (7) and (28) have sense depending on the problem under considerations." Let us try to explain what we mean.

Before going into details, it would be worthwhile to mention that the problem of random walk on the locally free group has some features of random walks in disordered materials. It is well known that the expectation values of a dynamic process in disordered system cannot be computed form equilibrium enumeration of all states of the given phase space, because some states are almost inaccessible during the time of the process.

We believe that the same is happened in our system: the local transition probabilities of the random walk on the graph $C(\mathscr{L}\mathscr{F}_n)$ are related to the effective coordinational number of the graph $C(\mathscr{L}\mathscr{F}_n)$ and depend on the way of consideration of the system: in "dynamical" approach we automatically count all accessible states (vertices of the graph) with appropriate weights computed along the true phase trajectories of the walker on the graph, while in "equilibrium" consideration we have many extra states because some enumerated vertices are "dynamically" almost inaccessible for the random walk (as it is explained below) for finite times of the process.

The computations of the expectation values for the random walk on the graph of the group $\mathscr{L}\mathscr{F}_n$ based on the exact enumeration of all primitive words imply (unevidently) the supposition that this target graph is symmetric. In other words, we supposed, without saying that clearly, that the number of μ-letter primitive words finishing by the generator with the number i is independent (or, at last, weakly dependent) on i. Only under such supposition the expectation values for the random walk on the graph can be computed on the basis of effective coordinational number $z = q(d) + 1$ (see Eq. (22)).

However, as it is shown below, only a small fraction of generators contribute to the main part of graph vertices. We can simply elucidate that computing the "correlation function" $G_n(i \mid d)$ which represents the portion of the primitive words of length is $\mu(\mu \to \infty)$ starting with arbitrary letter and ending with the letter i ($i \in [1, n]$). We have:

$$G_n(i \mid d) = \lim_{\mu \to \infty} \frac{V_n(\mu, i, d)}{V_n(\mu, d)} \tag{30}$$

where

$$V_n(\mu, i, d) = 2n + \sum_{s=1}^{\mu} 2^s \frac{(u-1)!}{(s-1)!\,(\mu - s)!} \mathbf{v}_{\text{in}}[\hat{T}_n(d)]^s \mathbf{v}_{\text{out}}(i) \tag{31}$$

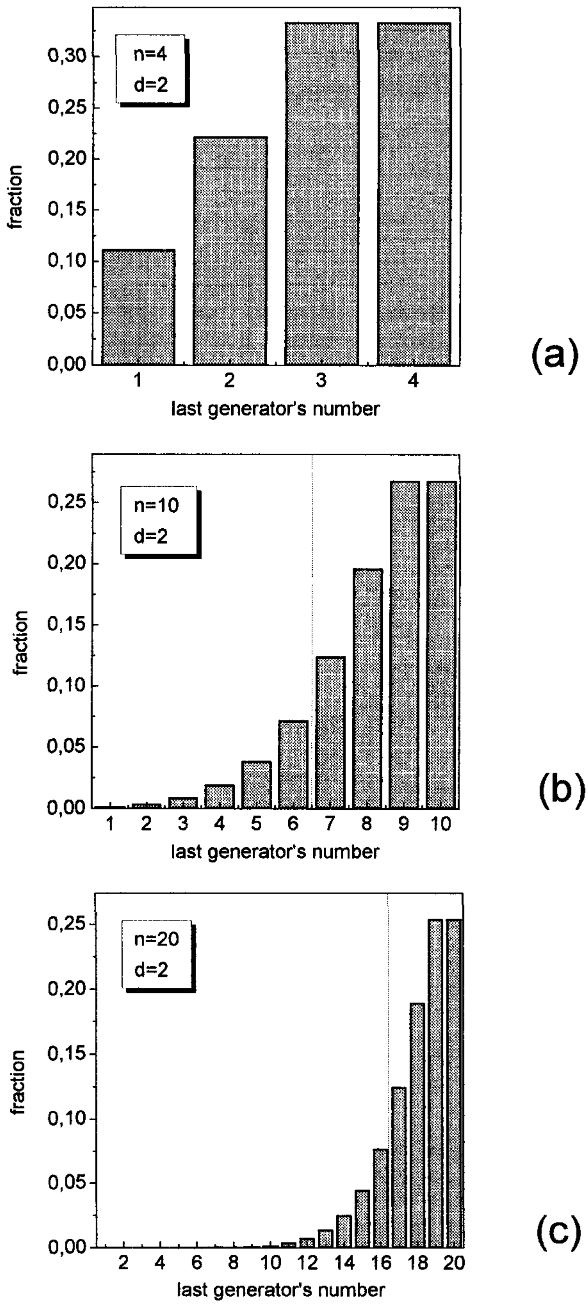


Fig. 3. Plot of the function $G_n(i|d)$ for $d=2$ and $n=4$ (a); $n=10$ (b) and $n=20$ (c).

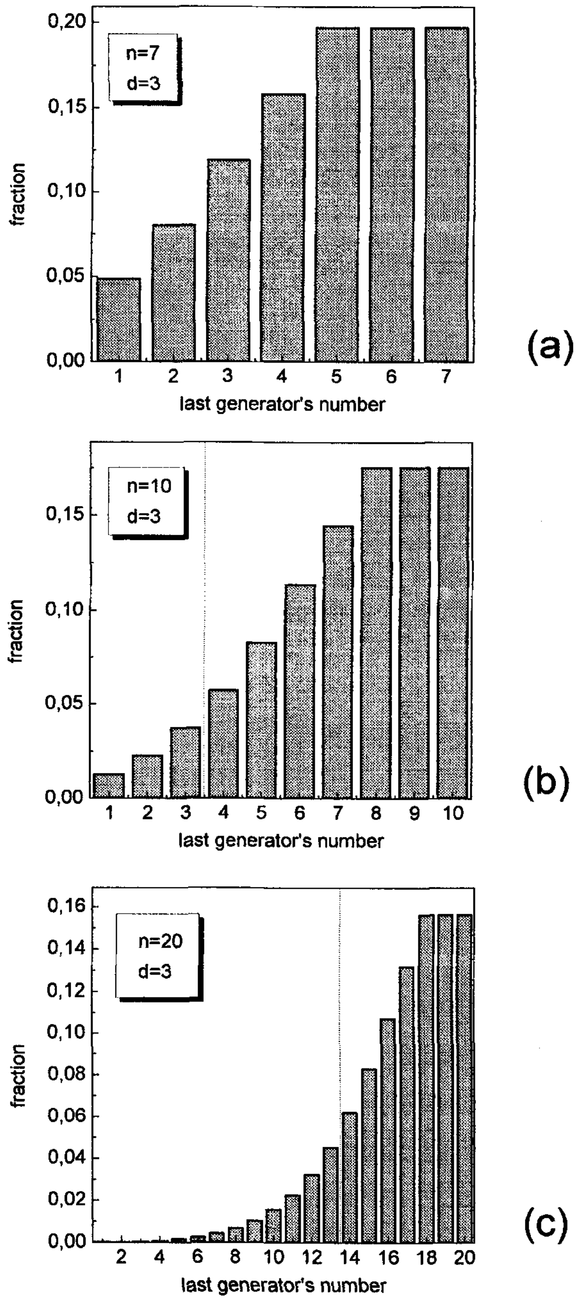


Fig. 4. Plot of the function $G_n(i|d)$ for $d=3$ and $n=7$ (a); $n=10$ (b) and $n=20$ (c).

and

$$\mathbf{v}_{\text{in}} = (1\ \ 1 \cdots 1) \qquad \text{and} \qquad \mathbf{v}_{\text{out}}(i) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \begin{matrix} \\ \leftarrow i \\ \\ \\ \\ \end{matrix} \tag{32}$$

(compare to Eqs. (18)–(21)). It is obvious that the following normalization condition is fulfilled:

$$V_n(\mu, d) = \sum_{i=1}^{n} V_n(\mu, i, d)$$

The results of calculations of the function $G_n(i\,|\,d)$ for $n = 4, 10, 20$ are shown in the Fig. 3a–c ($d = 2$) and for $n = 7, 10, 20$—in the Fig. 4a–c ($d = 3$). It can be easily seen that only about 4 generators (for $d = 2$) and 7 (for $d = 3$) give the main contribution to the ensemble of all primitive words of given length. Hence, our conjecture is as follows.

Conjecture 1. The coordinational number z_{cor} which governs the random walk on the graph of the group $\mathscr{LF}_n(d)$ should be extracted from the group $\mathscr{LF}_4$ for $d = 2$ and from the group $\mathscr{LF}_7$ for $d = 3$. In these cases only the corresponding graphs are almost symmetric, i.e., the distribution of words ending by the letter with the number i ($i \in [1, 4]$ for $d = 2$ and $i \in [1, 7]$ for $d = 3$) is almost independent on i.

The computation of the corresponding effective coordinational number z_{cor} using Eq. (22) gives us

$$\begin{aligned} z_{\text{cor}} &= 6 && \text{for} \quad d = 2 \\ z_{\text{cor}} &= 10.1 && \text{for} \quad d = 3 \end{aligned} \tag{33}$$

Of course, our consideration is rather crude and still is not supported by extended investigation of the structure of the graph of the locally free group. However we believe that the mentioned deviations between dynamical and statistical approaches reflect some nonergodic properties of random walk on the locally free group.

3. STATISTICAL APPROACH FOR WORDS ENUMERATION IN BRAID GROUP B_N

The problem of construction of an effective algorithm for enumeration of the words in the braid group B_n for $n > 2$ is one of very intriguing problems of the group theory.

In the present section we propose the approximate statistical approach for enumeration of all distinct primitive words in the group B_n for $n \gg 1$ which exploits some properties of locally free groups $\mathcal{LF}_n$ considered above.

The main idea is as follows. Take a look on the sequences of words in the braid group B_n from the point of view of the locally free group $\mathcal{LF}_n$ "with errors." To be more specific let us start with the following example:

Example 3. Write a random word W from the group B_7 consisting of 8 letters. Let it be for example:

$$W = \sigma_1^{-1}\sigma_4\sigma_5^{-1}\sigma_6^{-1}\sigma_5\sigma_1\sigma_6\sigma_2^{-1}$$

We reduce this word to the primitive one in two steps.

1. On the first step we act in the same way as in the case of locally free group $\mathcal{LF}_7$ and push all generators with smaller indices to the left *supposing that nearest neighbors do not commute at all.* We get:

$$W^{\text{reduced}} = \sigma_2^{-1}\sigma_4\sigma_5^{-1}\underbrace{\sigma_6^{-1}\sigma_5\sigma_6}_{\sigma_5\sigma_6\sigma_5^{-1}}$$

2. Now we can apply the Yang–Baxter relations for the triple $\sigma_6^{-1}\sigma_5\sigma_6$ and obtain after the cancellation of σ_5^{-1} and σ_5 the primitive word

$$W_p = \sigma_2^{-1}\sigma_4\sigma_6\sigma_5^{-1}$$

The first step of the contraction procedure completely coincides with what we have for the locally free group, while the second step we can regard (approximately, of course) as follows. If we have some pair, for instance, $\sigma_6^{-1}\sigma_5$, we can commute it *if* the nearest neighbor generator after σ_5 is σ_6. The probability to meet the generator σ_i in the Markov chain for the braid group B_n is of order of $p_{\text{err}} = 1/2n$. Later on we consider more general case taking p_{err} as the variational parameter.

In the Table 3 we show the results of our numerical simulations of the expectation value $\langle \mu \rangle / N$ for the random walk on the "locally free group with errors," $\mathcal{LF}_n^{\text{err}}(2)$, and compare them to the same value for the random walk on the braid group (see first columns of the Table 1).

We found asymptotically the perfect correspondence of the mean values $\langle \mu \rangle / N$ for the braid group and the "locally free group with the

Table 3

Groups	$\mathcal{LF}_n^{\text{err}}$ with 20% of errors [$p_{\text{err}} = 1/5$] $\langle\mu\rangle/N$	B_n $\langle\mu\rangle/N$
$n=5$	0.55	0.49
$n=10$	0.58	0.56
$n=20$	0.59	0.59
$n=50$	0.60	0.61
$n=100$	0.60	0.61
$n=200$	0.61	0.61

errors" for $p_{\text{err}} = \frac{1}{5}$. However the variance $\text{Var}(\mu)$ for the random walk on the group $\mathcal{LF}_n^{\text{err}}$ has the following scaling behavior

$$\text{Var}(\mu) \sim N^{2\nu(n)} \tag{34}$$

with $\frac{1}{2} < \nu(n) < 1$. The growth of the variance for the group $\mathcal{LF}_n^{\text{err}}$ we explain by the additional randomness produced by the "errors."

Thus we put forward the following conjecture:

Conjecture 2. The number of nonequivalent primitive words $V_n^{\text{braid}}(\mu)$ of length μ in the braid group B_n can be estimated as follows

$$V_n^{\text{braid}}(\mu) \simeq \langle V_n(\mu, d=2, p_{\text{err}}\rangle \tag{35}$$

where $V_n(\mu, d=2, p_{\text{err}})$ is the number of all distinct primitive words of length μ in the locally free group $\mathcal{LF}_n(2)$ "**with errors**" (we allow to commute the neighboring generators with the probability $p_{\text{err}} \approx \frac{1}{5}$) and the averaging is performed over the uniform probability distribution of "errors."

The mean number of different primitive words of length μ in the group $\mathcal{LF}_n^{\text{err}}$ we compute analytically in the forthcoming paper while below we present the main outline of these calculations.

It is easy to understand that the number of nonequivalent primitive words $V_n(\mu, d=2, p_{\text{err}})$ in the "locally free group with errors" can be calculated by means of averaging of Eq. (21) if we change slightly the matrix $\hat{T}_n$ replacing it by the random matrix $\hat{T}_n^{\text{err}}$:

$$\hat{T}_n^{\text{err}} = \begin{array}{c|cccccc|c} & f_1 & f_2 & f_3 & f_4 & \dots & f_{n-1} & f_n \\ \hline f_1 & 0 & 1 & 1 & 1 & \dots & 1 & 1 \\ f_2 & 1 & 0 & 1 & 1 & \dots & 1 & 1 \\ f_3 & 0 & 0 & 0 & 1 & \dots & 1 & 1 \\ f_4 & 0 & 0 & 1 & 0 & \dots & 1 & 1 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \vdots & \vdots \\ f_{n-1} & 0 & 0 & 0 & 0 & & 0 & 1 \\ f_n & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{array} \tag{36}$$

where above the diagonal we put everywhere "1" and below diagonal we have 1 subdiagonal filled by "1" or by "0" with probabilities $1-p_{\rm err}$ and $p_{\rm err}$ correspondingly: in all other places we have "0."

Thus, we arrive at the following final expression for the value $\langle V_n(\mu, d=2, p_{\rm err})\rangle$

$$\langle V_n(\mu, d=2, p_{\rm err})\rangle = 2n + \sum_{s=1}^{\mu} 2^s \frac{(\mu-1)!}{(s-1)!\,(\mu-s)!} \langle R_n(s, d)\rangle \tag{37}$$

where $\langle \cdots \rangle$ means the averaging over "the errors" with the distribution function of general form

$$\mathscr{P}_j^{\rm err} = \frac{\rm const}{n^\alpha}; \qquad (0 \leqslant \alpha < 1) \qquad \text{for} \quad j \in [1, n]$$

Let us mention that our arguments dealing with the words enumeration in the braid groups are based mainly on physical speculations supported by numerical simulations. The situation concerning the rigorous estimation of quantity $V_n^{\rm braid}(\mu)$ is as follows: from above $V_n^{\rm braid}(\mu)$ is bounded by the quantity $V_n(\mu, d=2)$ while the lower boundary is supposed to be $\langle V_n(\mu, d, 2, p_{\rm err})\rangle$. The proof of this last inequality will be published separately.

4. DISCUSSION

It has been pointed out in Section 1.2 that the identification of the top and bottom ends of each braid produces the knot (link). The topological state of linked paths can be roughly characterized by the "knot (link) complexity," η, extracted from rigorously defined algebraic knot invariants (see ref. 9 for details). Our further construction of mean-field theory of fluctuating entangled directed chain-like objects cannot be performed without knowing the entropy of ensemble of randomly generated closed braids of length N with fixed complexity η. On the basis of Magnus matrix representation of braid generators we found in ref. 9 that the link complexity η is strictly proportional to the length of the shortest irreducible (primitive) word, μ, written in terms of braid group generators. In Appendix B we reproduce from refs. 9 and 5 the brief description of the "knot complexity" construction.

Formally the problem of "knot (link) entropy" determination can be posed now as follows. Consider the ensemble of all knots produced by randomly generated braids of length N with fixed ends from the group B_n and suppose the value of knot (link) complexity, η, to be given. Due to the presence of topological constraints the entire phase space of knots (links) ensemble, Ω, splits into disconnected domains, $\omega\{\eta\}$, $(\omega \in \Omega)$ of "topologically similar" paths characterized by the link complexity η. The entropy of chains with given η one can write down formally as follows

$$S\{\eta\} \equiv \ln \omega\{\eta\} = \ln \sum_{\{\Omega\}} \delta[\eta\{W(\sigma_1,..., \sigma_n^{-1})\} - \eta] \tag{38}$$

where $W_N(\sigma_1,..., \sigma_n^{-1})$ is the particular realization of the N-letter word written in terms of braid generators.

Utilizing the relation $\eta = \mu$, we could conclude that Eq. (9) describes the distribution function of knots with fixed "complexity" obtained by randomly generated braids from the group B_n. The corresponding physical applications dealing with investigation of phase transition in the bunch of mutually entangled "directed polymers" will be published separately in II.

Let us briefly repeat now the main conclusions of the present paper.

1. The problem of the random walk on the braid group B_n is investigated numerically and the expectation values for the mean value and the variance of the primitive word are compared to the same values of the random walk on the locally free group $\mathscr{L}\mathscr{F}_n$.

2. The problem of random walk on the group $\mathscr{L}\mathscr{F}_n$ is considered analytically in two different ways: (i) using the direct approach dealing with "dynamics" of the words contraction in course of developing of the Markov chain on the locally free group, and (ii) by means of enumeration of all nonequivalent words of given length in the locally free group. It has been found that the dynamical consideration gives the results which being in perfect agreement with our numerical simulations, are in contradiction with the answers obtained by means of statistical approach. This conflict is partially resolved in Section 2.2 and is explained by the nonuniform structure of the graph corresponding to the locally free group. We believe that the discrepancy between statistical and "dynamical" approaches found here has the same origin as in the conventional disordered systems: some very unprobable configurations give the non-negligible contribution to the equilibrium partition function.

3. We propose a statistical method for enumeration the primitive words in the braid group B_n based on the consideration of "locally free groups with errors." The results of numerical simulations for the mean length of the primitive path of the random walk on the braid group and on the corresponding "locally free group with errors" with number of errors of order of 20% are in very good agreement. In our forthcomming publication[19] we bring some arguments in support of conjecture that the

number of rather long primitive words in the braid group is not sensitive to precise local commutation relations. We suppose to discuss the connection of the above-mentioned problems with the conventional random matrix theory, localization phenomena and statistics of systems with quenched disorder and to show the relation of the particular problems of random matrix theory to theory of modular functions.[19]

4. The computations of the entropy of entangled directed random walks performed in the present work should be applied for investigation of the ordering nematic-type phase transition in the bunch of entangled directed polymers under action of external field. We suppose to pay the main attention to construction of the simple mean-field Flory-type theory of interacting braided random walks with nonabelian topology in $1+1$ dimensions.

5. We believe that the problem of discovering the integrable models associated with the proposed locally free groups and developing the corresponding conformal field theory could help to establish the bridge between statistics of random walks on the noncommutative groups, spectral theory on multiconnected Riemann surfaces and topological field theory.

APPENDIX A

To calculate the eigenvalues of the matrix $\hat{T}_n \equiv \hat{T}_n(d=2)$ one has to solve the standard equation

$$\det[\hat{T}_n - \lambda \hat{E}_n] = 0 \tag{A.1}$$

where $\hat{E}_n$ is the unit $n \times n$ matrix.

Introduce minors $\hat{A}_n(\lambda)$ and $\hat{B}_n(\lambda)$ of the composite matrix $\hat{T}_n - \lambda \hat{E}_n$ as follows

$$\begin{gathered}\hat{A}_n(\lambda) = \begin{pmatrix} -\lambda & 1 & 1 & \cdots \\ 1 & -\lambda & 1 & \cdots \\ 0 & 1 & -\lambda & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \\ \hat{B}_n(\lambda) = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ 1 & -\lambda & 1 & \cdots \\ 0 & 1 & -\lambda & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\end{gathered} \tag{A.2}$$

Denote by $a_n(\lambda)$ and $b_n(\lambda)$ the determinants of matrices $\hat{A}_n(\lambda)$ and $\hat{B}_n(\lambda)$:

$$a_n(\lambda) = \det[\hat{A}(\lambda)]; \qquad b_n(\lambda) = \det[\hat{B}(\lambda)] \tag{A.3}$$

Now we can easily write down the set of recursion relations for $a_k(\lambda)$ and $b_k(\lambda)(0 \leqslant k \leqslant n)$

$$\begin{cases} a_k(\lambda) = -\lambda a_{k-1}(\lambda) - b_{k-1}(\lambda) \\ b_{k-1}(\lambda) = a_{k-2}(\lambda) - b_{k-2}(\lambda) \\ a_{k=1}(\lambda) = -\lambda \\ b_{k=1}(\lambda) = 1 \end{cases} \tag{A.4}$$

The solution of Eqs. (A.4) is

$$a_k(\lambda) = -\frac{\lambda + p_2}{p_1 - p_2} p_1^k + \frac{\lambda + p_1}{p_1 - p_2} p_2^k \tag{A.5}$$

where

$$p_{1,2} = \tfrac{1}{2}(-1 - \lambda \pm i\sqrt{(3-\lambda)(1+\lambda)}) \tag{A.6}$$

Taking into account the relation det $\hat{T}_n(\lambda) = a_n(\lambda)$, we can rewrite the equation for eigenvalues (Eq. (A.1)) in simplified form

$$a_k(\lambda)\,|_{k=n} = 0 \tag{A.7}$$

The solution of Eq. (A.7) is very straightforward

$$\tan\left\{n \arctan\sqrt{\frac{3-\lambda}{1+\lambda}}\right\} = -\frac{\sqrt{(3-\lambda)(1+\lambda)}}{\lambda - 1} \tag{A.8}$$

For $n \to \infty$ we get $\lambda_\infty^{\max} = 3$. To find the highest eigenvalue $\lambda_n^{\max}$ of matrix $\hat{T}_n$ for large $(n \gg 1)$ but finite n let us search the solution in the form

$$\lambda_n^{\max} = \lambda_\infty^{\max} - \varepsilon \qquad (\varepsilon \to 0) \tag{A.9}$$

Substituting (A.9) into (A.8) we find the expression for $\lambda_n^{\max}$ in terms of power series

$$\lambda_n^{\max} = 3 + c_1 n^{-1} + c_2 n^{-2} + O(n^{-3})$$

where we calculated two first terms of expansion: $c_1 = 0$ and $c_2 = -4\pi^2$.

The eigenvalues of the matrix $\hat{T}_n(d=3)$ can be calculated in the same way as of matrix $\hat{T}_n(d=3)$, although the computations are more tedious.

We can slightly modify the definition of the locally free group supposing that the new group depends on two parameters, c and d $(0 < c < d-1)$, and the commutation relations are as follows:

(a) Each pair (f_i, f_k) generates the free subgroup of the group if $c < |j-k| < d$;

(b) $f_j f_k = f_k f_j$ for $|j-k| \leqslant c$ and $|j-k| \geqslant d$

The methods developed for the locally free group can be easily extended to the problem of primitive words enumeration for this new group. The only modification concerns the structure of the matrix $\hat{T}_n(c, d)$. For instance, for $c = 1$ and $d = 3$ we have

$\hat{T}_n(c=1, d=3) =$

	f_1	f_2	f_3	f_4	$\dots$	f_{n-1}	f_n
f_1	0	1	1	1	$\dots$	1	1
f_2	0	0	1	1	$\dots$	1	1
f_3	1	0	0	1	$\dots$	1	1
f_4	0	1	0	0	$\dots$	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
f_{n-1}	0	0	0	0	$\dots$	0	1
f_n	0	0	0	0	$\dots$	0	0

APPENDIX B

It is known that the topological invariants of knots can be characterized by the algebraic polynomials. The most known are Jones, HOMFLY and Alexander invariants.[20] The Alexander invariants allow the following description.[21] Write the generators of the braid group in the so-called Magnus representation

$$\sigma_j \equiv \hat{\sigma}_j = \begin{pmatrix} 1 & 0 & \cdots & & \\ 0 & \ddots & & & \\ \vdots & & \boxed{A} & & \vdots \\ & & & \ddots & 0 \\ & & \cdots & 0 & 1 \end{pmatrix} \leftarrow j\text{th row}; \qquad A = \begin{pmatrix} 1 & 0 & 0 \\ t & -t & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{B.1}$$

Now the Alexander polynomial of the knot represented by the closed braid $W = \prod_{j=1}^{N} \sigma_{\alpha_j}$ of length N one can write as follows

$$(1 + t + t^2 + \cdots + t^{n-1})\, \nabla(t)\{A\} = \det \left[\prod_{j=1}^{N} \sigma_{\alpha_j} - e \right] \tag{B.2}$$

where index j runs "along the braid", i.e., labels the number of used generators, while index $\alpha = \{1,..., n-1, n,..., 2n-2\}$ marks the set of braid generators ("letters") ordered as follows $\{\sigma_1,..., \sigma_{n-1}, \sigma_1^{-1},..., \sigma_{n-1}^{-1}\}$. In our further investigations we repeatedly address to that representation.

Let us stress that in general the minimal irreducible length of the braid, introduced above, is not related directly to any topological knot invariants but we show below that nevertheless the "primitive word" can be served as a well defined characteristic of the "knot complexity". The "primitive word" has the simple topological sense which can be expressed in the following necessary condition. If the "primitive word" of some closed braid of n strings has the unit length then this braid belongs to the "trivial" class and the corresponding knot is represented by a set of n disjoint unentangled trivial loops uniquely.

We are interested in the limit behavior of knots or links invariants when the length of the corresponding braid tends to infinity, i.e., when the braid "grows." In that case we can rigorously define some more simple topological characteristics than the algebraic invariant which we call the "knot complexity."

Definition 1. Call the **knot complexity**, η, the power of some algebraic invariant, $f_K(t)$ (Alexander, Jones, HOMFLY) (see also refs. 9 and 5)

$$\eta = \lim_{|t| \to \infty} \frac{\ln f_K(t)}{\ln t} \tag{B.3}$$

Remark. By definition, the "knot complexity" takes one and the same value for rather broad class of topologically different knots corresponding to algebraic invariants of one and the same power, being from that point of view more weak topological characteristics than complete algebraic polynomial.

Let us summarize the advantages of the quantity introduced in Eq. (B.3) with respect to the corresponding topological invariant $f_K(t)$:

(i) One and the same value of η characterizes a narrow class of "topologically similar" knots which is however much broader than the class represented by the polynomial invariant $X(t)$. This allows one to introduce the smoothed measures and distribution functions for η.

(ii) The knot complexity η describes correctly, (at least from the physical point of view) the limit cases: $\eta = 0$ corresponds to "weakly entangled" trajectories while $\eta \sim N$ matches the system of "strongly entangled" paths. The later case has been discussed in details in ref. 5.

(iii) The knot complexity keeps all nonabelian properties of the polynomial invariants.

Our main goal in the paper concerns the estimation of the limit probability distribution $P(\eta, N)$ for the knots obtained by randomly generated closed B_n-braids of the length N. Let us stress that we essentially simplify the general problem "of the knot entropy." Namely, we insert an additional requirement that the knot should be represented by a braid from the group B_3 without fail.

ACKNOWLEDGMENTS

We are grateful to A. Comtet and S. Ouvry for useful discussions of the work and to Vik. Dotsenko for valuable critical remarks. S. N. would like to express his sincere thanks to A. M. Vershik with whom some related results have been obtained (see ref. 9) as well as to D. Parshin and M. Tsypin for important comments.

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